Introduction: Magic

A true mathematician who is not also something of a poet will never be a perfect mathematician.
(Karl Weierstrass, German mathematician, 1815-1897)

Mathematics and Poetry

> Poetry is the expression of the imagination. Reason respects the differences, and imagination the similitudes of things.
> (Percy Bysshe Shelley, English poet, 1792-1822, *Defence of Poetry*)

> In mathematics we look for similarities in differences and differences in similarities.
> (James Joseph Sylvester, English mathematician, 1814-1897) ditto

> The moving power of mathematical invention is not reasoning but imagination.
> (Augustus de Morgan, English mathematician, 1806-1871)

The great German mathematician David Hilbert (1862-1943) noticed one of his students had stopped attending his lectures. When he asked for the reason, he was told the student had left mathematics in favor of poetry. "Ah, yes," Hilbert said, "I always thought he didn't have enough imagination for mathematics."

Hilbert's derision of poets should be taken with a grain of salt. After all, he treated physicists similarly: he once declared that "physics is too hard for physicists." But Hilbert was not the only person who compared mathematicians to poets, in favor of the first. Voltaire, for example, said that "there was more imagination in the head of Archimedes than in that of Homer." Even poets made the same claim: the American poet Edna St. Vincent Millay entitled one of her sonnets "Euclid Alone Has Looked on Beauty Bare."

Every mathematician can attest, from his personal experience, to the closeness between mathematics and poetry. In today's society, which is so dependent on scientific progress, mathematics directly or indirectly influences every aspect of our life. But its main attraction, for the professional as for the amateur, lies not in its applicability but in its beauty.

There is a secret in this. What can be alluring in a logical argument? How does the cold and abstract mathematical world resemble art? What does geometry have in common with music, or arithmetic with poetry? In this book I want to touch upon this secret. I will try to find the attribute shared by mathematics and poetry that arouses a sense of beauty. But before we can embark upon unraveling this mystery, we must answer two questions, What is "mathematics"? and What is "poetry"? – two highly non-trivial questions. We will examine both questions from the same angle – that of beauty. What is it that makes a mathematical argument or a poem beautiful? And why are the sensations of beauty aroused by both so similar?

"What is beauty" is one of those eternal riddles that should be approached with a sense of trepidation. But hopefully, our approach, of comparing the appearance of beauty in two different fields, will provide some handle on the problem. The relatively small overlap between mathematics and poetry narrows down the space in which the answer is to be sought for.

In search of hidden structures

> This poem is about people;
> What they think and what they want
> And what they think they want.
> Beyond this, there aren't many things in the world
> Deserving our attention.

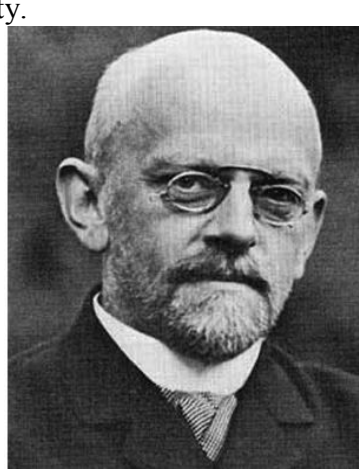
(Nathan Zach, "Introduction to a Poem," from *Different Poems*)

The citations above, from mathematicians and poets alike, point to a common denominator of mathematicians and poets: they both hunt for hidden patterns. On the other hand, their hunting grounds are clearly far apart. If there is affinity between poetry and mathematics, it is not to be found in their subjects. Poetry is concerned with people: what they think, want, sense, and feel. Mathematics, in contrast, is indifferent to emotions. Its topics come from the exact sciences, so in final account it searches for order in the material world.

"Order" is indeed a key concept for understanding beauty, and it will be the subject of the first part of the book. But the concept of order alone will not bring us far in our quest. While useful for understanding the beauty of mathematics, its role in poetry is only secondary. The core of the similarity between mathematics and poetry lies elsewhere: in their thinking mechanisms. This similarity will be illustrated in the second part of the book. In it, I shall list techniques and themes that are common to poetry and mathematics, with an attempt to unearth the resemblance in the ways they give birth to beauty.

Finally, in Part III, after having gained insights into the mechanisms governing these two areas, I will return to the riddle of beauty on a more theoretical level.

But before we dive in, I would like to provide you with a little taste of what is to come. In the next two chapters, I will try to illustrate the main ingredient in mathematical and poetical beauty: magic. The sense of beauty, so we shall learn from these examples, is the outcome of a sleight of hand played by the poem or the mathematical argument, which prevents us from understanding them in their entirety.



David Hilbert, a German mathematician, 1862 – 1943
Possibly the last mathematician that mastered all major mathematical fields of his time

A First Example: Displacement

Dream Mechanisms

The first technique we will examine, displacement, is not unique to poetry or to mathematics. It appears in almost every area of human thought. The discoverer of this mechanism, who also coined the name, was Sigmund Freud, the founder of psychoanalysis.

Throughout his life, Freud regarded his first major work, *The Interpretation of Dreams*, as his crowning achievementIn it he developed his view of the human psyche, that can be summarized in that the psyche is not monolithic. It is inhabited by different, and at times conflicting, forces. Reuben Fine, a chess grandmaster who left chess in the 1940s to become a psychoanalyst, formulated this tersely: Psychoanalysis, he claimed, is simply "dynamic psychology." That is, the study of the different forces struggling against one another in the same mind.

The basic premise of *The Interpretation of Dreams* is that a dream is the expression of an unconscious wish. The wish is often forbidden and, under normal circumstances, repressed. The task of repression is entrusted to a part of the psyche that Freud called the "censor." It ensures that the Pandora box of the unconscious remains closed. All this, however, is true only during wakefulness. At night, when the watchdogs of the psyche are asleep, the box's lid slides open a bit, allowing some of the forbidden contents to surface. Even at night, however, they are not allowed to appear as they are, and they must disguise themselves to avoid being identified. Otherwise, Freud argued, we would be alarmed and awaken.

The major part of *The Interpretation of Dreams* is devoted to the thought mechanisms employed for the camouflage. For example, "inversion": instead of appearing directly, an idea may be represented by its opposite. Pursuit will be represented in a dream by flight, stripping by dressing. Another camouflage technique is "condensation": several ideas are compressed into one. A single image in a dream can simultaneously represent the dreamer's father, teacher, and even his son. When this happens, we find it difficult to follow what is happening, which serves the dream's purpose of concealment.

A third mechanism, which is the subject of this chapter, is "displacement" - the diversion of attention from an important factor to one that is less significant. The main character of the play is shunted aside to the murky edges of the stage, while the spotlights focus on some other, minor figure. The main idea is thereby presented as if incidentally, away from the focus of attention. This mechanism appears also in in mathematics and in poetry, where it usually generates a sense of beauty.

Displacement in a poem

The poem "About Myself" by the Israeli poet Lea Goldberg is ars poetica, meaning that it is about poetry making. The conclusion that the poet draws about the role poetry plays in her life is painful:

> [...]
> My images are
> Transparent like windows in a church:
> Through them
> One can see
> How the light of the sky shifts
> And how my loves

Fall
Like dying birds.
(in: *Lea Goldberg: Selected Poetry and Drama*, trans. Rachel Tzvia Back)

Among the various stratagems of the poem, the most transparent is the metaphor. In fact, a second-order one, that is, a metaphor within a metaphor: the poems are compared to images, while the images, in turn, are likened to church windows. But although finely-crafted and effective, these metaphors are not the core of the poem. The true strength of the poem is in its three last lines. "I live in poems," the poetess declares, "while in the real life my loves fall dead" - a complaint that accompanied Goldberg her entire life. Moreover, she hints at a causal connection, that the loves die because of the poems. The birds are not only seen through the windows, it is implied that they also smash against them.

This is a striking confession, but sincerity by itself does not produce beauty, and if the message had been delivered directly, the poem would not be as moving. The last lines, with the blow they deliver straight to the belly, owe their effect mainly to the fact that the reader is unprepared. And this is achieved by the ploy of incidental statement . The despair is seemingly expressed offhandedly. The poems-windows, and the birds-loves seen through them, as if serve only to witness the transparency of the windows. Thus, the painful message of the dead loves is delivered casually, pretending that they are a mere illustration of something else. This is displacement.

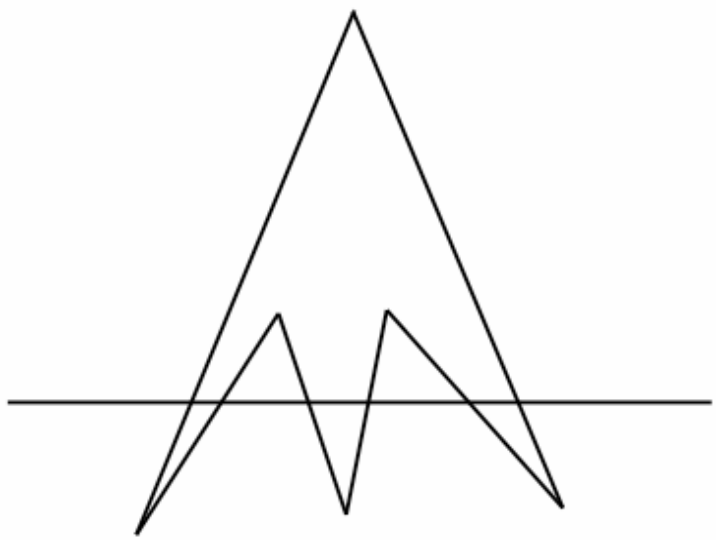
Judging by this example, displacement achieves the same effect in poetry as in dreams: distracting one's attention. Thanks to this technique, the message bypasses the guards of the mind. When one's attention is sidetracked, the message is absorbed subliminally, evading direct confrontation. This is a way to touch but not really, like a feather's touch.

Lea Goldberg (1911-1970); born in Kovno, Lithuania; immigrated to Palestine in 1935

When a straight line meets a polygon

In mathematics, and in science in general, change of perspective is frequently the key to the solution of a problem. But the aim is different from that in poetry or dreams: it is not .meant to disguise the message, but to cast things in a new light. However, there is similarity in one aspect: beauty. In both fields the contribution of displacement to the sensation of beauty is the same – it prevents explicit understanding of the message. Something remains not entirely comprehended.

Here is an example. Look at the six-sided polygon - a hexagon – in the picture. It is not convex, that is, its sides "cave in." In the picture there is also is a straight line, crossing all six sides of the hexagon.



A six-sided polygon (a hexagon), with a straight line that intersects all its sides.

Question: Can you also draw a seven-sided polygon (a heptagon), with a straight line intersecting all its sides?

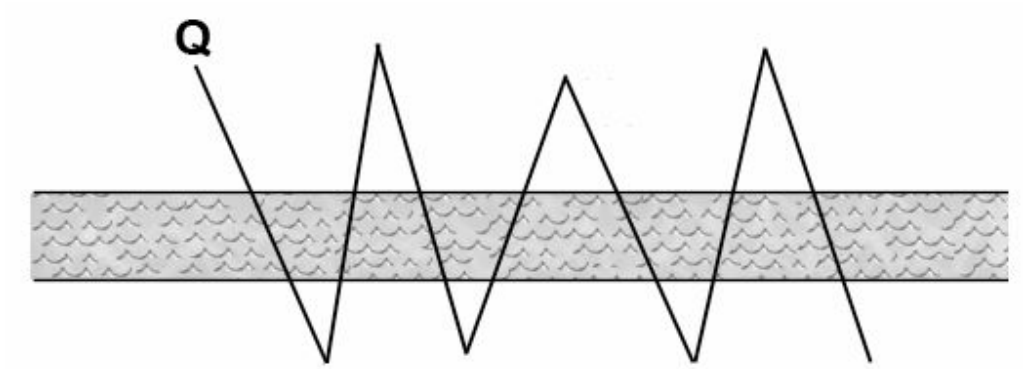
When a mathematician asks you to perform a task, be wary. Chances are that he's having fun at your expense, and that the mission is impossible. And so it is in this example – trying a bit (I recommend doing it -) will quickly convince you that it cannot be done. Why? One way to see this uses change of perspective. The question begins with a polygon, and asks for a straight line meeting all its sides. Instead, begin with the straight line, and trying to draw the polygon only later.

Before presenting the solution, let me formulate the principle on which it is based. It is called the "river crossing rule." When a person crosses a river one time, he ends up on the other bank; after two crossings, he is back on the first bank; and after three crossings, he is on the other bank again. The rule is that an even number of crossings brings you back to the original bank, and an odd number of crossings brings you to the other side.

According to this rule, I could know, for example, whether I went through the door of my office an even or odd number of times (my office is located on the sixth floor, and I can't enter or leave it through the window). I don't know what this number is, but I am certain that it is even: every time that I entered, I also went out (these lines are being written outside the office). This is simple, but simplicity is not to be confused with shallowness. The river crossing law is at the heart of some profound mathematics.

Now, let's return to our straight line and heptagon. As mentioned, we will begin by drawing the straight line, and only afterwards the heptagon. In order to draw the heptagon, we will start from one of its points - call it Q - and draw the edges one by one. How many times will we cross the straight line on the way? Seven times, of course, since each of the seven sides crosses the straight line. Since 7 is odd, by the river crossing rule the heptagon must end on the other side of the straight line, not on the side on which it began. That is, it ends on the side opposite Q. This, however, is impossible, since the heptagon is a closed polygon, and should end at Q. This contradiction means that the assumption that every side of the heptagon intersects the

straight line is impossible.



After seven crossings, we are on the other side of the river, so we couldn't have drawn a closed polygon

In fact, the change of order is not essential: the argument could be formulated also starting with the heptagon. The straight line should cross each side of the heptagon once, so it crosses the boundary of the heptagon seven times. This means that it goes in-out-in-out-in-out-in, ending inside the heptagon. But this is impossible – the line must end up outside. But the river crossing formulation seems to me more natural.

How Many Games Are There in a Cup Tournament?

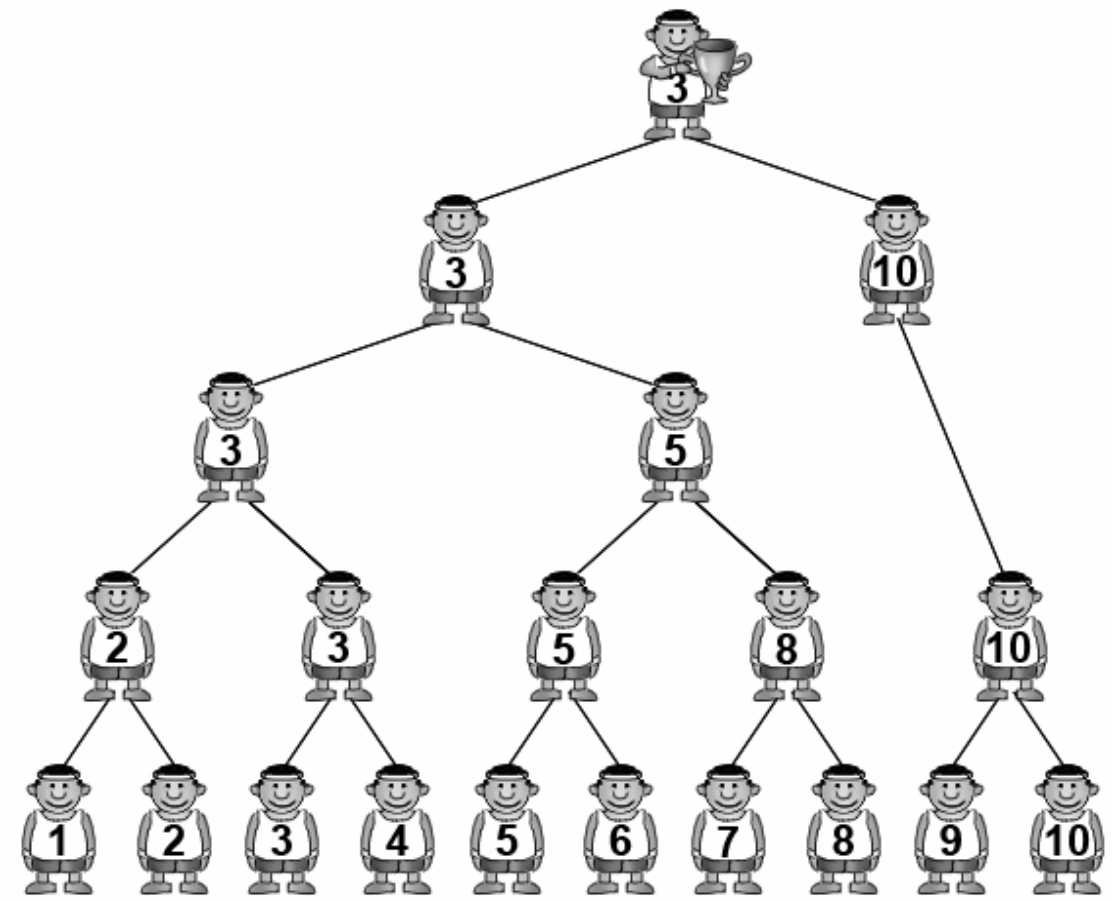
A cup tournament is a competition in which the players are paired off, each pair plays a game, and the winner advances to the next round.

Question: How many games will be played in a tournament with 16 players?

Let's look at the rounds. In the first round, the 16 players are arranged in 8 pairs, 8 games are played, and the 8 winning players go on to the next round. The 8 players in the second round are put in 4 pairs, and will play 4 games. In the third round, there will be 2 games, and in the fourth, that will determine the cup holder, only 1. All in all 8 + 4 + 2+ 1 = 15 games were played.

The number 16 is a power of 2 - it is 2^4, that is, $2\times2\times2\times2$. Hence, in each round all the players can be paired off. But a tournament can also be held with a number of players that is not a power of 2. In such a case, in some rounds there will be an odd number of players, and they cannot all be paired off. When this happens, the players are paired off except for a single player, and the extra player advances to the next round without playing. How many games will be held then?

There is one secret that mathematicians share with poets, to which we will return again and again: they think in examples. The way to abstraction goes through examples. And the simpler the example, the better. There is no such thing as a "too simple example." The simplest example here is that of a single player. When there is only one player, the number of games is 0. In the next simplest example, a competition with 2 players, there is one game. When there are 3 players, 2 games are held: a game between a pair, followed by another game between the winner of the first round and the third player. Let's advance a bit further, and consider a tournament with 10 players. In the first round, 5 games will be played, and 5 players will advance to the next round. The continuation is presented in the next picture, which shows that in this case there are $5 + 2 + 1 + 1 = 9$ games.



In the first round 5 games were played; in the second, 2; in the third, 1; and in the fourth, 1
Altogether 9 games.

This was not very hard, but in the case of 1000 players, the calculation will be exhausting. Is there an easier way to calculate the number of games? Note that in all these cases, the number of games equaled the number of players, minus 1. This is probably no coincidence. It must be a rule: **the number of games is smaller by 1 than the number of players**.

A good conjecture is the decisive stage in every discovery, but it requires proof. In this case, the proof requires seeing the picture from another angle: from looking at the winners to looking at the losers. To illustrate this, let's examine the case of a tournament with 1000 players. Each of the 999 players who did not win the cup lost exactly once: there was exactly one game in which he dropped out of the tournament. Since each game has exactly one loser, in order for 999 players to lose, there have to be 999 games! Much simpler than the tedious round by round calculations.

Why do we have a sense of magic about this solution? Because everything has happened too fast. Just as in the Lea Goldberg poem, the message was slipped under our noses. Even after we understand the solution, something remains mysterious.