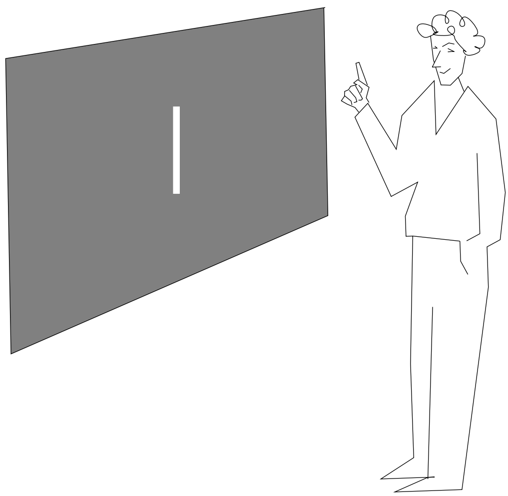


Part 1

Elements

What is Mathematics?

The book of nature is written in the language of mathematics.

Johannes Kepler, Astronomer

The Queen of Sciences

Mathematics is the queen of sciences and arithmetic the queen of mathematics.

Carl Friedrich Gauss, Mathematician

In second grade classes, I try to explain the importance of numbers. I tell the children a story of a king who hated numbers so much that he forbade their use throughout his kingdom. Together we try to imagine a world free of numbers, and discover that life in it is very limited. Since it is forbidden to mention a child's age, children of all ages enter the first grade; you cannot pay for your groceries, nor can you set up an appointment, since you are not allowed to mention the number of hours and minutes.

This is only an illustration of the importance of mathematics in our lives. As civilization and technology advance, our lives become more and more dependent on mathematics. Steven Weinberg, a Nobel laureate in physics, dedicated two chapters in his book, "Dreams of a Final Theory," to subjects beyond physics: mathematics and philosophy. He writes that time and again he is surprised to discover how useful mathematics is, and how futile is philosophy.

To understand why this is so, one must understand what is mathematics. This is not a simple question — even professional mathematicians find it difficult to answer. Bertrand Russell said of mathematicians that they "don't know what they are doing." (His judgment of philosophers was even harsher: A philosopher in his eyes is "a blind man in a dark room looking for a black cat that isn't there.") This is true in at least one sense: Most mathematicians do not bother to ask themselves what it is, exactly, that they are doing.

To try answering this question, we will start with a simple example: What is the meaning of $3 + 2 = 5$?

In the first grade, I ask the children to examine how many pencils there are when you add 3 pencils to 2 pencils. They know that "addition" means "joining." Therefore they join 3 pencils and 2 pencils and count: 5 pencils. Now I ask, "How many buttons are there when you add 3 buttons and 2 buttons?" "Five buttons," they answer, without missing a beat. "How do you know?" I insist. "We know from the previous question." "But the previous question was about pencils. Maybe it's different with buttons?"

They laugh. But not because the question is pointless. On the contrary. It contains the secret of mathematics: abstraction. It does not matter if the objects in question are pencils, buttons or apples. The answer is the same in all cases. This is why we can abstractly say: $3 + 2 = 5$.

This is an elementary, but characteristic, example: Mathematics abstracts thinking processes. Obviously, every thought is abstract to a certain degree. But mathematics is unique in that it abstracts the most elementary processes of thought. In the example of $3 + 2 = 5$, the process involved is the joining of objects — 3 objects and 2 objects. One can ask many questions about these objects: What are they — pencils or apples? Are they in your hand or on a table? And if they are on a table, how are they arranged? Mathematics ignores all these details, and asks a question that relates not to the various details, but only to the fact that these are objects that are joined: the resulting amount. That is, how many objects are there?

Abstract thought is the secret of man's domination of his environment. The power of abstractions lies in the fact that they enable us to cope efficiently with the world. In other words, they save effort. They enable going beyond the boundaries of the "here and now" — something discovered here and now can be used in another place and at another time. If 3 pencils and 2 pencils equal 5 pencils, the same will be true for apples, and it will also be true tomorrow. A one-time effort provides information about an entire world.

If abstractions in general are useful, then all the more so is mathematics, which takes abstractions to their limit. Therefore, it is not surprising that mathematics is so useful and practical.

Should Everyone Learn Mathematics?

People, on learning that I am a mathematician, often react with a thin smile, barely hiding a grimace of agony: "Mathematics wasn't one of my strong subjects." For so many people learning mathematics is such a tormenting experience, that each generation asks the same question — what for? Why is this torture necessary? Shouldn't most people just give up on the attempt to learn mathematics? Nowadays, when a calculator can instantly perform mathematical operations — what is the point of learning the multiplication table, or long division?

One answer is that mathematics is the key to all professions demanding knowledge of the exact sciences, and there are many of those these days. But mathematics is important not only for understanding reality. It offers much more than that — it teaches abstract thought, in an accurate and orderly way. It promotes basic habits of thought, such as the ability to distinguish between the essential and the inessential, and the ability to reach logical conclusions. These are some of the most significant assets that schooling can provide.

The question still remains unanswered — why is it so difficult? Must mathematics be a cause of suffering? A currently popular answer is "no" — the problem lies in the teaching. Common opinion is that many children considered to be "learning disabled" are actually "teaching disabled." But it can't be that simple. Blaming the teachers is too simplistic, and unreasonable. Anyone who claims that for hundreds and thousands of years mathematics teachers have been doing a bad job, must explain why this is so — and why it isn't so in other subjects.

The special problem in teaching mathematics lies in the difficulty of conveying abstractions. You can tell people the name of the capital of Chile, but you can't abstract for them. *This is a process each person must accomplish on his or her own.* One must mentally pass through all the stages from the concrete to the abstract. The teacher's role in this process is to guide the student, so that he experiments with the principles in the correct order. This is not a simple art, and it is not easy to come by. But neither is it impossible. One of the purposes of this book is to relay some of the principles along the path of such "midwifery" teaching, as Socrates put it.

The Three Mathematical Ways of Economy

I didn't have time to write you a short letter, so I wrote a long one.

Blaise Pascal, Mathematician

Mathematics is being lazy. Mathematics is letting the principles do the work for you so that you do not have to do the work for yourself.

George Pólya, Mathematician

The true virtue of mathematics (and not many know this) is that it saves effort. This is true of any abstraction, but mathematics has turned economy of thought into a form of art. It has three ways to economize: order, generalization and concise representation.

Order

Carl Friedrich Gauss was the greatest mathematician of the 19th century. One of the most famous stories in the history of mathematics tells of how his talent came to light when he was 7 years old. One day, his teacher, looking for a break, gave the class the task of summing up all the numbers between 1 and 100. To his surprise, young Carl Friedrich returned after a few minutes, or perhaps even seconds, with the answer: 5050.

How did the seven-year-old accomplish this? He looked at the sum he was supposed to calculate, $1 + 2 + 3 + \cdots + 98 + 99 + 100$, and added the first and last terms: 1 and 100. The result was 101. Then, he added the second number to the one before last, that is, 2 and 99, and again the result was 101. Then, 3 and 98, again 101. He arranged all 100 terms in 50 pairs, the sum of each equaling 101. Their sum was thus 50 times 101, or 5050.

What little Gauss discovered here, was order. He found a pattern in what seemed to be a disorganized sum of numbers, and the entire situation changed — suddenly matters became simple.

Imagine a phone-book arranged by a random order, or an unknown order. To find a phone number, you would have to go through each and every name. The order introduced into the phone-book, and the fact that we are familiar with it, saves a great deal of effort. A relatively small effort invested in alphabetical arrangement is returned many times over.

Or, think how much easier it is to live in a familiar city than in a strange one. A local knows where to find the supermarket or the laundromat. Knowing the order of the world around us provides us with orientation. Science, and mathematics in particular, has taken upon itself to discover the order of the universe, so that we may adjust our actions to it.

Generalization

There are many jokes about the nature of mathematics and mathematicians, but the following is probably the best known of them all. I make a point of telling it to my students in every course I teach, since it is not only the most familiar, but also the most useful. It illustrates the principle of mathematical practice: Something once done does not require redoing.

How can you tell the difference between a mathematician and a physicist? You ask: Suppose you have a kettle in the living room; how do you boil water? The physicist answers: I take the kettle to the kitchen, fill it with water from the tap, place it on the stove and light the fire. The mathematician gives the same answer. Then you ask: Suppose you have a kettle in the kitchen; how do you boil water now? The physicist says: I fill the kettle with water from the tap, place it on the stove and light the fire. The mathematician answers: I take the kettle to the living room, and this problem has already been solved!

This brings economizing of thought *ad absurdum*, by placing it before true economization.

"This has already been solved" was also the answer we heard from the children who said they did not need to check how many buttons are 3 buttons and 2 buttons, since they had done the same with pencils. It appears, whether overtly or hidden, in each mathematical proof, and in every mathematical argument. "We have already done this, and now we can use it." In fact, this idea lies behind every abstraction: What we discover now will also be valid in other situations.

Proof in Stages: Induction

There is a mathematical process that is based entirely on the principle of "this has already been done." It is called "Mathematical Induction." A certain point is established in stages, with each stage relying on its predecessor, that is, on the fact that the previous case "has already been solved."

We will encounter this process several times throughout the book, but will not mention it explicitly. For example, the decimal system is inductive: First, ten single units are collected to equal a new entity called a "ten"; then ten tens are collected to equal a new entity called a "hundred," and so forth. Yet another example is calculations: All algorithms used to calculate arithmetical operations are based on induction.

Concise Representation

The third mathematical economy is in representation. We are so used to the way numbers and mathematical propositions are represented, we forget that methods of representation were not always so sophisticated, and it was not so long ago, relatively speaking, that mathematical notation was much more cumbersome.

Let's begin with the representation of numbers. Up until about three thousand years ago, numbers were represented directly — "4" was represented by four markings, for example four lines. This is a good idea for small numbers, but impractical for larger ones. Using the decimal system, we can now represent huge numbers concisely: A "million" only requires seven digits.

The second type of economy is in the representation of propositions. A "mathematical proposition" is the equivalent of a sentence in spoken language. Up until a little over two thousand years ago, mathematical propositions were phrased in words, for instance: "Three and two is five." Then, a very useful tool was invented: the formula. Its originator was probably Diophantus of Alexandria, who lived during the 3rd century B.C. Formulas are not only shorter, they are also more accurate and uniform, and allow systematic handling.

Historical Note

The notation we currently use developed slowly and gradually. Its current form was only established during the 16th and 17th centuries. The sign of equality (=), for example, only appeared in the mid-16th century. Its inventor, the Englishman Robert Recorde, explained his choice by saying that "no two things can be more equal than a pair of twin lines of one length."

Mathematical Economy

Mathematics has three ways of saving effort:

— *Order:* Discovering a pattern makes orientation easier.

— *Generalization:* A principle discovered in one area can be applied to another area.

— *Concise Representation:* The decimal system is a wonderfully economic way of representing numbers; mathematical formulas represent propositions in a brief and clear manner.

The Secret of Mathematical Beauty

Euclid alone has looked on beauty bare.

Edna St. Vincent Millay, Poet

If the solution is not beautiful, I know it is wrong.

Buckminster Fuller, Architect and Inventor

In one second grade class, I showed the children an elegant way of proving the commutative rule of multiplication (we will encounter it in the chapter about the meaning of multiplication). A child sitting in the first row looked ahead for a moment, and then said quietly: "It's beautiful."

Ask a mathematician what it is about his profession that appeals to him, and nine out of ten times the answer will be — "beauty." Mathematics is useful in everyday life, but to those who deal with it that is not the essence. To them its main characteristic is its beauty. A mathematical discovery rewards its owner, and those who study it, mainly with aesthetic satisfaction. What does mathematics have to do with beauty? What possible relation could there be between the cold and dry subject of mathematics and the beauty found in art?

This brings us to the notoriously difficult and intricate question, "What is beauty?" It may actually be mathematics, the uninvited guest in this arena, that can shed light on the answer. For it is there that we find a pretty unanimous agreement on the meaning of beauty: A mathematical idea is beautiful when it introduces a new and unexpected element, one that appears as if from nowhere. The person who invented the decimal organization of numbers undoubtedly felt a powerful sense of beauty. The first person to discover the possibility of summing numbers by writing them one above the other surely had a sense of aesthetic satisfaction.

The Great Book in Heaven

The famous Hungarian mathematician, Paul Erdős, used to talk of the "Great Book in Heaven," containing the most elegant proof for every theorem. I believe many would agree that the first page of The

Book should be dedicated to Euclid's proof that there are infinitely many prime numbers. Not only is it one of the most elegant proofs known in mathematics, it is also one of the most ancient.

A *prime number* is a number that is not divisible by any other integer (that is, integral number), except for 1. For example, 2 is a prime number, as are 3 and 5. The integer 4 is not prime, as it is divisible by 2. Any integer can be written as a product of prime numbers (not necessarily different ones: 4, for instance, is 2 times 2). The first five prime numbers are 2, 3, 5, 7 and 11.

Of course, 11 isn't the last prime number. For example, 13 is a larger prime number. But Euclid claimed that he could prove there was a prime number greater than 11, without knowing what it was! The idea is simple. Look at the product of the first five prime numbers, namely, $2 \times 3 \times 5 \times 7 \times 11$. The result, 2310, is obviously divisible by 2, 3, 5, 7 and 11. Therefore, the integer 2311, the result of adding 1 to 2310, is not divisible by any one of these numbers (if an integer is divisible by 2, its successor is not divisible by 2; if it is divisible by 3, its successor is not divisible by 3, etc.). But, like any other integer, 2311 is divisible by some prime number, and, as mentioned before, it cannot be 2, 3, 5, 7 or 11 since 2311 is not divisible by these. Therefore, there must be a prime number that is not one of these numbers.

The same will obviously apply if we take the first 100 prime numbers. What we demonstrated is that for each prime number there is a greater prime number — and therefore there are infinitely many prime numbers!

This proof is doubly elegant. First, because of the idea, appearing as if "from nowhere," of multiplying all prime numbers up to a given integer and adding 1; second, because the indirect proof demonstrates the existence of a greater prime number without actually naming it.

"Knowing Without Knowing"

We still haven't arrived at the root of the matter — what inspires a sense of beauty? In art, for example, beauty isn't necessarily derived from the introduction of unexpected ideas! Is there a relationship between mathematical beauty and the beauty of poetry, for example, or that of music?

To understand this relationship, consider the power of poetic metaphors. The beauty of a metaphor is derived from its indirect message, something said without actually being said, so that the receiver does not have to look it straight in the eye. Take for example the following metaphor, from the Song of Songs:

> *Look not upon me, that I am swarthy, that the sun hath tanned me; my mother's sons were incensed against me, they made me keeper of the vineyards; but mine own vineyard have I not kept.* (Chapter 1, Verse 6).

The metaphor in the last line contains a simple message, but for a moment the reader can pretend not to understand it, as if it is truly about a vineyard that is not well guarded.

What happened here? As in mathematics, an "idea from somewhere else" suddenly appeared — a vineyard instead of an erotic message. As a result, we perceive the message on one level, while not completely absorbing it on another level. This is "to know without knowing." Likewise, in a surprising mathematical solution the unanticipated connection of ideas enables us to understand the newly discovered order on one level, while the ordinary tools of reason, still using the old concepts, do not suffice to grasp it.

Is this true of all types of beauty? I believe so. Rare beauty is wondrous in our eyes — in other words, it contains something that we do not fully understand. For example, a magnificent view inspires a sense of beauty because it lies beyond the scope of our ordinary tools of perception.

Mathematics and Art

> *It is true that a mathematician who is not also something of a poet will never be a perfect mathematician.*
>
> Carl Weierstrass, Mathematician

Mathematics has two characteristics in common with art: One is order, and the other is economy and concision. Art, like mathematics, finds order in the world. Music, for instance, is organized noise, while paintings create order in the visual experience. As for concision — poetry, for example, is famous for shortening and compressing many ideas into one saying. In German, the word for poetry is "Dichtung," meaning "compression." The poet Ezra Pound defines great literature as *language charged with meaning to the utmost possible degree.*

These are all related to "knowing without knowing." Order inspires a sense of beauty when it is perceived by the subconscious. There are two possible explanations for such a mode of perception: Either the order is so surprising that standard perception does not keep up with it, or it is too complex to be perceived by reason. Economy and concision also have the effect of "knowing without knowing" — the idea flies past us so quickly that we do not have time to grasp it. The same is true of the compression of several meanings into one expression — it does not enable conscious comprehension of all the significations.

The elementary arithmetic we learned as children contains some of the most beautiful mathematical discoveries ever made. Why, then, is it not perceived by most people as beautiful? Mainly because it is often learned mechanically, in a way that does not reveal its beauty. But it is not too late, and those who are able to see the principles of elementary mathematics in a new light will be able to rediscover their beauty. I can testify that this is what happened to me.

Layer upon Layer

Professors demonstrated free thought / And thoughts of gymnastic instruments, in groups and individually./ But most of their words remained unclear to me, / I probably was not yet ready.

Yehuda Amichai, "I Am Not Ready", from *"Poems"*

Fermat's Narrow Margins

In 1637, the French mathematician, Pierre Fermat, wrote a conjecture in the margins of a book, a copy of Diophantus' *Arithmetica*: "I have discovered a truly marvelous proof," he added, "but the margin is too narrow to contain it."

Generations of mathematicians were tormented by the thought that the proof of what became the most famous mathematical conjecture of all times was truly lost, and dedicated themselves to reproducing it. After a while, it became clear that Fermat, like many others that followed him, was deluding himself, and was actually mistaken in his proof. When a proof was finally found in 1995, by the Englishman Wiles, it became obvious that it could not have fit into the margins of a book. It was 130 pages long, and if added to the many arguments on which it was based, would fill thousands of pages.

Shorter proofs than the one to Fermat's conjecture are also constructed of many layers, each one based upon the other. Each layer is established in turn, and serves as a basis for the next, according to the principle "this has already been done." There are other fields in which knowledge is built on previous knowledge, but in no other field do the towers reach such heights, nor do the topmost layers rely so clearly on the bottom ones.

The first fact one must know about mathematical education is that this is true not only of advanced mathematics, but also of elementary mathematics. There, too, knowledge is established in layers, each layer relying on the preceding one. The secret to proper teaching lies in recognizing these layers and establishing them systematically.

A famous anecdote in the history of mathematics tells of the impossibility of shortcuts. The hero of the story is Euclid, who lived in

Alexandria between 350 and 275 B.C. and authored "The Elements," the most influential geometry book of ancient times – possibly of all time. Among other achievements, he defined in it the terms "axiom" and "proof," two of the greatest achievements of mathematics. Ptolemy, the king of Egypt at that time, asked for his advice on an easy way to read the book. "There is no royal road to mathematics," replied Euclid. Even kings cannot skip stages.

Note: The 5th century Greek historian, Stobaeus, attributes the same story to different characters — Alexander the Great and his teacher, Menaechmus.

The same is true of elementary mathematics. However, since it deals with the bottom of the tower, the number of layers it establishes is smaller. There are no long chains of arguments as in higher mathematics. This is one of the reasons it is appropriate for children. In another sense, though, it is harder. Some of its layers are hidden and difficult to discern, as if they were built underwater, in a place that is difficult to view. Noticing them requires perceptive observation. They are easy to miss and skip. Elementary school mathematics is not sophisticated, but it contains wisdom. It is not complex, but it is profound.

Mathematics Anxiety

Education researchers use the term "mathematics anxiety." There is no history anxiety, or geography anxiety, but there is mathematics anxiety. Why only mathematics?

The main reason lies in its layered structure: Mathematics anxiety arises when one stage is unheedingly skipped. As mentioned before, many of the layers of mathematical knowledge are so elementary that they are often easy to miss. And when this happens, and an attempt is made to establish a new layer on top of the missing one, neither the teacher nor the student can discern the origin of the problem. The student hears something that is meaningless to him, since he is "probably not yet ready." The teacher is also perplexed, since she cannot identify the source of the difficulty. When one does not understand the origin of a problem, unfocused fear arises and anxiety is born.

A "layer" needs not be an explicitly stated piece of information. Sometimes it is the acquisition of experience. For example, to acquire the concept of the number, one must have extensive experience in counting. Something happens in the mind of a child when he is counting. Something is gradually built, requiring investment of time and effort even if the results are not immediately apparent.

One cannot mention mathematics anxiety without also mentioning the other side of the coin — the joy of mathematics. Just as anxiety is not associated with any other subject, so too the happiness that beams from a child's face when he understands a mathematical principle is not seen in any other subject. As likely as not, there is a connection between the two.

An Example of the Importance of Not Skipping Stages

Here is an example from my personal experience of what happens when you skip a stage. An experienced teacher would probably not have fallen into the trap as I did. She would have known how difficult the term I was trying to teach, that of "greater by ... than ..." or "more by ... than ...," is for children. But that trap turned out to be an instructive lesson for me. I learned the importance of establishing concepts in the proper order, and how far one can go when this is done.

For a certain period of time, I taught two first grade classes in Maalot. One day, nearing mid-year, I arrived with the intention of teaching both classes problems which included expressions such as "greater by 4 than ..." or "4 more than ..." In the first class, I wrote on the blackboard: "Donna has 4 pencils more than Joseph. How many pencils does Donna have, if Joseph has 5 pencils?" The order in which the data were presented was not incidental. I deliberately began with the relation between the number of pencils Donna and Joseph had, and not with the absolute number of pencils Joseph had. I wanted them to understand that it is possible to discuss the relation between the numbers without knowing the point of reference (how many Joseph has).

Up to that point, the children had no difficulty translating real life stories into arithmetical expressions. This time was an exception. Confusion prevailed in the classroom. I tried to phrase the question directly: "Joseph has 5 pencils, Donna has 4 more pencils. How many pencils does Donna have?" But this didn't help either. Most of the children were not following.

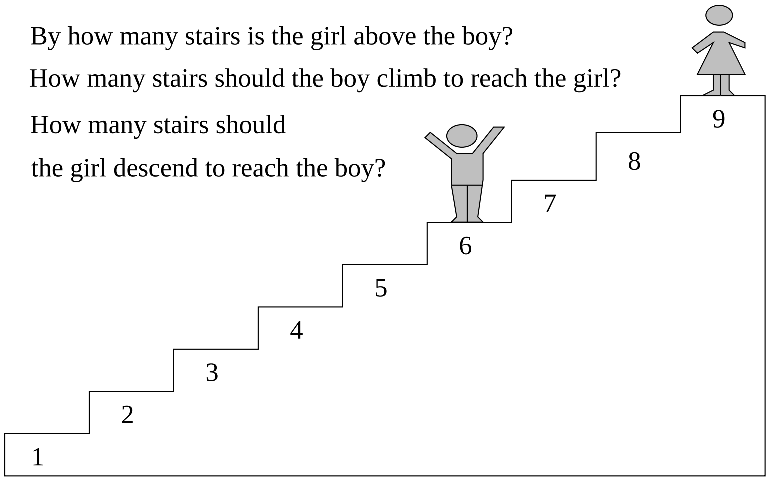
By that time I understood that I had skipped a stage. As a matter of fact, I had skipped more than one. It was not only the concept of a relation between two elements that are not yet known, that was difficult for the children. It was also the concept of *having 4 more than* or *greater by 4 than* itself that were unfamiliar, and a first-grade teacher should have been well aware of that fact. This is not a concept the child encounters in his everyday life. Most children are familiar with the term "greater than" but not necessarily with the meaning of "greater by 4."

There was no other way but to start anew. I drew a large circle and a square on the blackboard, and asked the children to draw an equal number of triangles inside the square and the circle. Then I asked them to add one triangle to the circle, and asked which shape contained more triangles. Then I asked how many more. At this point the lesson ended.

From this lesson I moved on to the other first-grade class. By now I was wiser, and began the lesson properly, from the beginning. I invited two children to the blackboard and gave each 5 crayons. I asked which child had more, and was told that they both had the same number. I gave one child an extra crayon and asked: Who has more now? How many more? By how many does the other child have less? I gave the first child yet another crayon and repeated the same questions. I continued to give the child more crayons, and asked at each stage how many more crayons he had than the other child, and by how many the other child had less. Then I gave the second child more crayons, one by one, until they both had an equal number. The next stage was going in the opposite direction — taking away one crayon after another from one child, and asking at each stage who had more, and by how many more. Throughout the entire process, without giving in, we also asked who had less, and by how many less.

I then drew a set of stairs on the blackboard and numbered them. I drew two children — one on stair 9 and one on stair 6. I asked the class how many stairs the child positioned lower needed to climb to reach the child on the higher stair. Then, how many stairs the higher child needed to descend to reach the lower one. I asked: "By how many stairs is the first child higher up than the second?" and "By how many stairs is the second child lower than the first?" We went through several similar examples.

Now we took the abstraction one step further. Instead of the concrete stairs, we switched to age differences. I asked one child by how many years he was older than his brother. By 3 years, he answered. By how many years was his brother younger than him? From then on, we had a ball; I asked: How old he would be when his brother was 20 years old? And how old his brother would be when he was 100 years old? And when he was a 1000 years old? Some of the children followed me through to the high numbers — in almost every first grade class there are students who can calculate in the hundreds, even complicated calculations like $1000 - 3$. Of course, they were very amused by the thought of what would happen to them in a hundred or a thousand years.



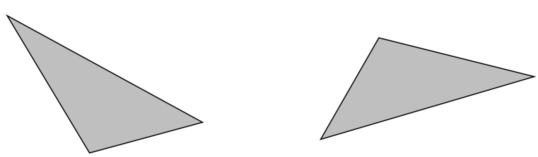
The final part of the lesson was dedicated to personal experimentation. We used improvised abacuses: wooden skewers on which we threaded beads made of play-dough. (These are better than regular beads since the children make them themselves and can feel the material in their hands. The play-dough beads also don't roll noisily on the floor.) The children paired up, and I asked that in each pair one child would put three more beads on the skewer than the other child. This was another source for enjoyment. I didn't tell them how many beads each child should thread, nor who should have more beads. I only mentioned the difference, and this drove them to higher numbers. In one pair, for example, one child threaded 10 beads, confident that he would be the winner. When he found out he was beaten since his partner had threaded 13 beads on her abacus, he added 6 beads to his. A well known pedagogical dictum is that a lesson should go through three stages: from the concrete, to drawing, and finally to abstraction. In this sense, the lesson was exemplary: We began with the concrete (the crayons). Then we moved on to the drawing of the stairs, and finally on to the discussion of the ages of siblings, in which the numbers were not concretely presented. Finally, we ended the lesson with an active implementation of the concepts we had learned (play-dough beads and wooden skewers).

So much for didactics. What about content? What conceptual structures did the children acquire during this lesson? More than first meets the eye. First of all, they learned the concept of relation, like "greater" or "smaller," between numbers. Furthermore, they understood the message I was trying, unsuccessfully, to convey in the first class: that we can discuss a relation between numbers without having the actual numbers at hand. The assignment they received at the end of the lesson was to create a situation where one child had three beads more than his or her partner, without being told explicitly how many beads each child should have.

In addition, the children learned that a relation can be viewed from both directions and that matters look different from each angle: If one is greater by three, than the other is smaller by three. They realized that the relationship changes when you change one of its components. (If I have three more, you can have the same if you add three to your own, or if I subtract three from mine.)

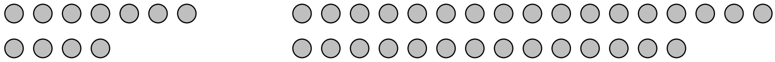
Of course, they also learned the concept of "greater by ..." itself, which was the purpose of the lesson. And right along with it — the relationship between "greater by ..." and addition: If you add 4 to a number, the result is greater by 4 than the original number.

Another idea conveyed in this lesson was the law of conservation. "Conservation" means that something remains constant while other things change. For example, when you rotate a triangle, its position changes, but the relationship between its sides remains constant; the



A conservation law: The angles are preserved when the triangles are rotated

angles are the same. This lesson taught conservation of difference: The difference between the ages of two brothers is always the same. If you are 7 years old and your brother is 4 years old, the difference between your ages is 3. Twenty years from now, the difference will remain unchanged: 27 is greater than 24 by 3. That is, if you increase the numbers by the same amount, the difference between them will remain constant. This law will accompany the children throughout their school years. They will encounter it, for instance, when expanding fractions, where the ratio between two numbers does not change if you multiply them by the same number.



Law of conservation: 17−14=7−4

* * *

The first of the two lessons always serves me as a reminder of how elusive are the structures of thought that are established at an early age; how easy it is to believe that they exist in children's minds, and how easy to forget that even the most elementary structures need to be established at some point, through hard work.

But it was the second lesson that was truly instructive, and there I learned a lesson which will reappear throughout this book. I realized that by establishing concepts in the proper order, and teaching them through concrete experimentation, you can go a long way. The concept of "greater by ..." was established through concrete experimentation, dwelling upon each of its aspects, even those that seemed most simple. We insisted on explicit phrasing, even for the most obvious principles. Through these we managed to go far beyond what I had dared to even dream of at the beginning of the lesson.

"Ask Me an Easier Question"

As a fresh parent, I used to ask my children mathematical questions. I do it less these days — asking questions is not a natural pattern for a parent-child relationship. But one day, when I asked my youngest daughter a question, she taught me an important lesson: "Ask me a simpler question," she said. She was not trying to evade the question, but asked me to provide her with a preliminary step. If a question is hard, it usually means that some previous stage is missing.

I use this phrase in my classes. When students find a problem too hard, I tell them the story of my daughter, and encourage them to do the same: Whenever they fail in solving a hard problem, they should ask for a simpler one. My aim is to make them aware of the possibility that there is a stage missing in their knowledge. This does not only avert frustration, but also makes them aware of their thinking processes — an important goal by itself.

Step by Step

One day I was watching a small class of weaker students. Some of them had a hard time calculating sums like 8+6. They were studying conversion between hours and minutes. The teacher asked them the following question: There were 3 meetings of 50 minutes each. How much time did the meetings take altogether, in terms of hours and minutes? I knew well what would have happened if free discussion were to follow: One or two students would know the answer, the rest would be left behind. So I asked the teacher's permission to step in; I told the students that in mathematics one has to go very slowly, and then I told them the story of my daughter (who was about their age) and her "give me a simpler question" request. I said that I was going to ask them questions in steps, and promised that each question will be simple. Then I asked a girl who beforehand refused to answer questions, how many hours are in 60 minutes. This she knew — one hour. Then I asked how many hours and minutes there are in 61 minutes. That was not hard, either. Then I asked about 62 minutes, and 63, not skipping a stage. This was slow, but all kids were with me, and all had a sense of achievement. When we got to 90 minutes, I dared to make a jump — and now, how about 100 minutes? And 110? Then we went slowly again: 111, 112, until we got to 119 minutes — an hour and 59 minutes. Then we got to 120 minutes, which they said, as I had expected, is one hour and 60 minutes. But what are 60 minutes, I asked, and they got to the desired answer — 120 minutes is two hours. From there on to the number appearing in the original question, 150 minutes, it was easy and all the students knew the answer. Everybody can make forward steps, if they are small enough. One only has to know how to break the problem into small steps, and how not to be in a rush. In the long run, going in small steps saves time, not wastes it.

Divide and Conquer

An error no less common than skipping stages is teaching two ideas (or more) at once. Ideas should be taught separately, even if they are not dependent on each other, and even if the order in which they are taught is immaterial. It is important to teach each stage individually. *Divide and Conquer,* or in other words, "break the principles into components," is one of the principal rules of good teaching.

In many cases, breaking a problem into stages is all that is required — the child will do the rest himself. For example, when a child finds it difficult to calculate 2×70, it is sometimes enough to ask him how much is 2×7 and he will complete the missing information on his own. It is enough to provide a person with an intermediary step on the ladder, and he will climb it himself. This reminds me of a saying I once heard from a teacher of mine: "A mathematical proof is a non-trivial combination of trivial arguments" — the difficult part is breaking a problem into stages, and not each separate stage in itself.

Whole Numbers

God made the whole numbers;
all else is the work of man.
Leopold Kronecker, Mathematician

Why were Numbers Invented?

My 9-year-old daughter likes to sign birthday cards with the words: "With lots and lots and lots ... of love," filling half the page with "lots."

It could be that thousands of years ago, before numbers were invented, this method was used as a substitute for counting. Instead of saying "3 stones" the caveman would say: "stone, stone, stone."

Now it is clear why numbers were invented: to economize! Instead of filling up half a page, my daughter could write "a hundred times lots of love."

Though I must admit it doesn't have the same effect.

* * *

In one of my third grade classes, I wanted the children to understand how economic the use of numbers is. So, I told them a story. "The story I am about to tell you," I warned them in advance, "happened before something was invented. Can you guess what that something is?"

A caveman returned to his cave after a day of hunting, and said to his wife: "I brought you a rabbit, a rabbit, a rabbit and a rabbit." His wife answered: "Thank you, thank you and thank you."

The children had no trouble guessing: The story took place before the invention of numbers. Nowadays we would say in short "4 rabbits," or "3 thank yous" (or, as is more customary, *many thanks*).

There is no problem saying "rabbit, rabbit, rabbit and rabbit" when there are 4 rabbits. But just imagine what would have happened if the caveman had brought 100 rabbits! Numbers save a lot of work, utilizing all three ways of mathematical economy: representation, generalization (the number 4 is used to count rabbits, pencils and cars) and order — knowing how many items there are of each kind provides essential information about the world, and creates a certain order.

The numbers, 1, 2, 3, ... , were born of the fact that the same type of unit can be repeated several times. Since their invention was so natural, they were given the name "natural numbers." All other types of numbers were invented at a later stage, and they are indeed less natural, since they are more distanced from real life: fractions, negative numbers, real numbers, complex numbers, and the list goes on.

In this book, when the word "number" appears on its own, it refers to natural, namely whole numbers.

Why do Numbers Play Such a Central Part in Mathematics?

Ask a passerby what is mathematics, and the answer will probably include the word "numbers:" Mathematics deals with numbers. Professional mathematicians know this isn't accurate. Certain mathematical fields, such as geometry, do not deal directly with numbers. Still, there is much truth to the popular view: Numbers do indeed have a special role in mathematics. They appear in almost every mathematical field, at least indirectly. Why is this so?

Mathematics, as mentioned before, abstracts the elementary processes of thought. Numbers play such a central part in it because they are a result of the abstraction of the most elementary of all processes: sorting the world into objects. Man recognizes a part of the world, separates it from the rest, defines it as a single unit, and gives it a name: "apple," "chair," "family." This is how words were created, and how the number "1" was born — "one apple," "one chair," "one family." Natural numbers came about through repetition of a unit of the same kind: "2 apples, 3 apples ..."

Numbers with a Denomination and Pure Numbers

"I planned on having one husband and seven children, but it turned out the other way around."

Lana Turner, Movie Star

Numbers are important, explains Lana Turner. But first and foremost is the denomination, namely, what the number counts.

The concept of the number did indeed begin with numbers that have denominations, that is, with the counting of objects. The abstract number came along at a later date, born of the fact that important properties of numbers, such as the results of arithmetical operations, are not dependent on the denomination: $2 + 3 = 5$ is true of apples and chairs alike. Therefore, man abstracted: From 4 apples and 4 chairs he invented the "pure" number, that is, a number without a denomination: 4. Pure numbers can be used to phrase propositions that are true of any object.

It is the concrete examples that lead to abstraction, and therefore numbers with denominations should be taught before pure numbers. In other words, the number should be taught through counting existing objects. A first grader should count as much as possible. This is the only way to establish the concept of the number. A first grade classroom should be full to the brim with buttons, beads, Popsicle sticks, straws. When counting, the denomination should always be mentioned. The answer to the question: "How many pencils do we have here?" is not merely "4," but "4 pencils."

First Lesson on Denominations

The following is a suggestion for a first lesson on denominations. Warn the class that you are about to give them strange instructions. When you have their attention, ask one of the children: "Give me two." The ensuing discussion will teach the children that when you say "two" you must also say two of what.

Sets

The first arithmetical operation is neither addition nor subtraction, but that which gave birth to the number — the definition of a unit. In other words, it is the separation of an object from the rest of the world, giving it a name and identifying it as "a unit," which can then be repeated several times.

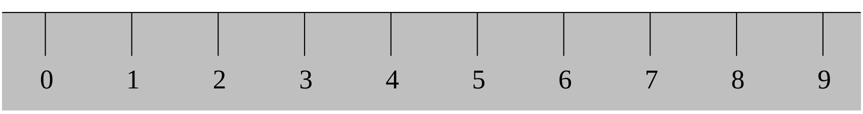
In fact, the definition of a unit is more basic an operation than the invention of numbers. It serves to establish other mathematical concepts as well. In particular, it is the basis for a most important mathematical concept, second only to the number: the concept of the set. This concept originated in the finding that a few items can be grouped to form a new unit, called a "set." Several people group together to form a set called a "family," and 5 players form a basketball set (or team).

This operation is most prominent in the decimal system used to organize numbers. Ten terms are grouped to create a new unit called a "ten." Ten tens can then be used to create another unit, a "hundred."

Order

The numbers are used not only for counting, but also for ordering objects: “first, second, third, . . . ” Numbers can be compared, and ordered according to this comparison. When a number a is smaller than a number b, we write: $a < b$. For example, $3 < 5$. Some children find it difficult to remember which direction the sign goes, and most textbooks provide a mnemonic: The larger, open side of the sign “$<$” faces the larger number. The sign “$<$” is used only for pure numbers: We do not use the notation “3 apples $<$ 5 apples.” Therefore, in first grade, when the numbers are mainly used with denomination, it is better to use the terms “more” and “less,” or the more official term “greater than” and “smaller than,” and leave the “$<$” notation to the second grade.

There is a basic tool for visualizing the order among numbers. It is called the “number line.” It is a straight line, with the numbers marked on it at evenly spaced intervals. Again, this is used for the order among pure numbers, and therefore its study is better deferred to the second grade. First graders are still in the counting stage, and the even spacing, as well as the association with the straight line, and the identification of the numbers with points, are too abstract for them.



Meaning and Calculation

Arithmetic in Search for Meaning

Most people associate "arithmetic" with arithmetical operations, and arithmetical operations with their calculation.

The first association is definitely correct. Arithmetic does indeed deal mainly with arithmetical operations. In contrast, the second association is far from being true. The operations and their calculation are not one and the same. Calculation is only the second stage. First and foremost is understanding the meaning of the operations.

The meaning of an operation lies in its link to reality: which situations require addition of numbers, which require subtraction and which multiplication or division. For example, the meaning of addition is to join: $3 + 4$ is achieved by joining together four objects and three objects. The meaning of subtraction (amongst others) is to remove: $7 - 3$ corresponds to a situation where 3 out of 7 objects are removed.

It may sound simple, but in fact therein lies most of the depth of arithmetic, mainly because the rules that guide the use of operations are derived from their meaning.

Arithmetical Play-Acting, Drawings and Stories

As with any abstract concept, the road to the meaning of operations begins with the concrete. To understand addition, the child must experience joining sets of objects over and over. To understand division, he must experiment in dividing a set of objects into equal sets.

There are three ways to experience the arithmetical operations: play-acting, drawings and stories. An especially efficient way of initially introducing an arithmetical operation is arithmetical play acting. At first, the teacher is the director. Two children are invited to the front of the classroom; one is given 3 straws and the other 2 straws — how many do they have altogether? After the children reply "5 straws," point to the set of 3 straws, the set of 2 straws, and say out loud: "3 plus 2 equals 5." Let's write it down. But remember — arithmetic likes to be concise. Instead of words, we use signs — and now write the plus sign on the blackboard: $2 + 3 = 5$. Then, a different variation: One child holds 4 pencils in his hand and the other holds 3 pencils. The first child gives all his pencils to the second child: How many pencils does the second child now have? What is the appropriate mathematical expression? (The term in general use is "sentence," or "mathematical sentence.") How many pencils does the first child now have? The same play can be repeated, only this time the second child gives all his pencils to the first.

In the next stage, the children become the directors. They decide on the arithmetical play, play in it and write the mathematical sentences on the blackboard.

Then come the arithmetical drawings. The teacher can demonstrate by drawing 3 flowers alongside 2 flowers — how many are there altogether? Then the children should draw on their own. First on the blackboard, and then in their notebooks or personal boards. For addition, the drawings are simple; they become more complex with subtraction and then multiplication and division. Incidentally, here is a "tip" for subtraction drawings: When you demonstrate, for instance, $5 - 2$ by drawing 5 balloons and removing 2 of them, don't erase the 2 you remove, but just cross them out — this way the subtracted amount can still be seen.

The last stage is that of telling arithmetical stories, using words. "Zack has 3 flowers, Amber has 2 flowers. How many do they have altogether?" This is already abstract, because the numbers are not represented by objects or drawings, but by their names.

Inventing Your Own Arithmetical Stories

I hear — I forget. I see — I remember. I do — I understand.

Confucius

You can't learn how to drive by watching others drive, and you can't learn to dance by watching "Swan Lake." Similarly, the meaning of the operations cannot be fully understood just by hearing arithmetical stories from others. You must be able to invent such stories on your own.

The ability to do so is the true test of understanding the meaning. Tell a story that matches the expression $3+4$, or $4-3$ ("Dina received 4 arithmetic exercises as homework. She solved 3. How many does she have left?"), or 3×4, or $12 \div 3$.

Inventing arithmetical stories has an additional advantage: It teaches the reversibility of processes. First, we made the transition from an arithmetical story (Amber has 4 arithmetic exercises, she solved 3, how many does she have left?) to an arithmetic exercise $(4 - 3)$. Now we learn that the opposite is also possible: Given the exercise $4 - 3$, a matching story can be invented. In fact, more than one story.

Calculation Means Finding the Decimal Representation of the Result

What is "calculation?" Figuring out the result of an exercise, of course. But this is only a partial answer, which does not touch on the main point. For the last thousand years most of mankind has been representing numbers using the decimal system, and hence the essence of calculations is in figuring out the decimal representation of the result.

Before the decimal system was invented, calculators had a simple life. The number 4, for example, was represented by $||||$. The meaning of a calculation such as $8 + 4$ was to draw 8 sticks, and 4 more alongside them. The result was also written using the same marking, like this:

$$|||||||| + |||| = ||||||||||||\,.$$

Is there any sophistication to such calculations? None at all. The caveman did not need to send his children to school to learn this. All the knowledge required here is that the meaning of addition is "to join" — the result is achieved by joining the two sets of lines. It does not require any calculation.

Nowadays the same exercise is written differently: $8 + 4 = 12$. Is there any cleverness in this? Is there any point in learning this in school? This time the answer is a decisive "yes." This requires a true operation: the grouping of a ten. Of the 12 lines in the result, ten are grouped into one ten. This calculation provides information: The result includes one ten and two ones. Something has been said here about *the decimal representation of the result.*

Calculation is figuring out the decimal representation of the result from the decimal representation of the problem's components. This is one of the reasons why knowing how to calculate is so important in elementary school. Its purpose isn't just to figure out the results of problems, but also to achieve a deeper understanding of the decimal system.

How do We Calculate?

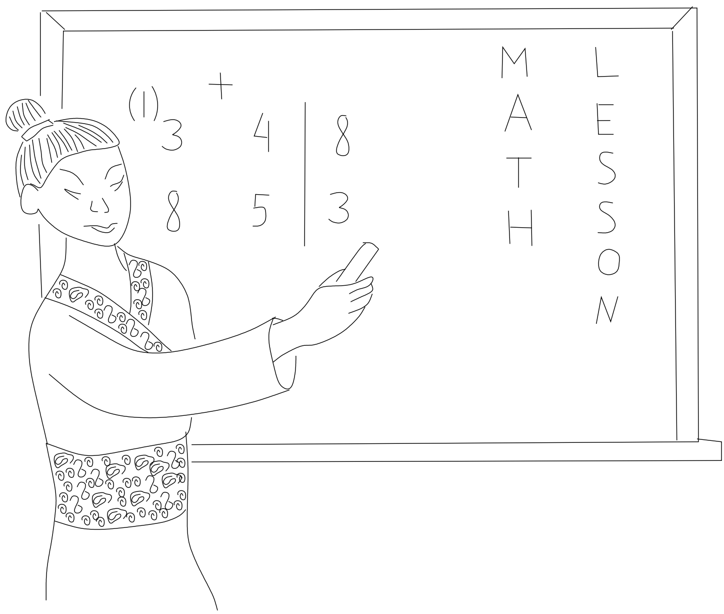
Most people remember from their arithmetical studies mainly the ways to calculate the operations. These are recipes for performing the steps, one by one, according to a predetermined order — much like a recipe for baking a cake. A fixed recipe, dictating actions in a certain order, is called an "algorithm."

The algorithms we were taught in school are so deeply ingrained in us that we tend to forget they are not exclusive. That is, there is more than one way to perform each arithmetical operation. The algorithms we are familiar with, those taught in school, make use of a pen and paper and are based on writing the number vertically, one above the other. Hence their names: vertical addition, subtraction and multiplication. Division received a different name — "long division." These algorithms were introduced in Europe by the Arabs during the 12th century, and were established slowly. The fact that these algorithms remained unchanged for hundreds of years, to this day, bears witness to their wisdom.

Historical Note

The word "algorithm" is derived from the name of a 9th century mathematician, Al–Khwarizmi, who was born in the region of Khwarizm, now part of Uzbekistan, and lived in Baghdad. The Europeans learned the decimal system, and methods of performing operations with it, from the translation of his book. Therefore, they called any calculation an "algorism," which out of confusion with the Greek word "arithmus" (number) later became "algorithm."

Our generation's calculation algorithms make use of a pen and paper. This was not always so. Between the 12th century, when the decimal system was introduced in Europe, and the 16th century, there was a competition between the "algorithmists," those who calculated using paper (as we do today) and the "abacusists," those who used an abacus. The decimal system became standard only after the algorithmists prevailed.



Why are Addition and Subtraction Exercises Written Vertically?

Ever since the first grade I have known that Hebrew is written on lined paper and arithmetic on graph paper. What I didn't understand then was why. This should be explained to the students — it isn't a state secret. The purpose of the squares is to make it easier to write the ones beneath the ones, the tens beneath the tens and so forth. What for? So that we can add or subtract items of the same kind: ones with ones, tens with tens. It is simply a matter of a "common denomination." Take, for example, the exercise:

$$\begin{array}{r} 23 \\ +64 \\ \hline 87 \end{array}$$

The vertical writing makes it easier for us to understand that the 3 ones in 23 should be added to the 4 ones in 64, and the 2 tens in 23 to the 6 tens in 64.

The Decimal System

Ten fingers have I
They can build anything.
Rivka Davidit, “Ten Fingers Have I”

Organization and Representation of Numbers

Anyone using large numbers needs a good system to organize and represent them. The system used today is the decimal system. This is undoubtedly one of the most ingenious and useful inventions ever made by man. It enables not only concise writing of numbers, but also simple and efficient calculation. It is our generation, the computer generation, that provides the decisive proof of its usefulness. Much effort is currently invested in researching ways to simplify calculations. If a more efficient method of representing numbers existed, it would probably have been discovered by now. The fact is that the principles of number representation in computers are still identical to the principles of the decimal system.

Two Principles: Grouping Tens and Place Value

The decimal system is based on two principles. One relates to the organization of numbers, or more precisely, of object sets. The other relates to the writing of numbers.

The first principle is about collecting — grouping ten elements to form a new unit. This process is repeated: Ten ones are collected to form a ten, ten tens to form a hundred, ten hundreds a thousand, and so forth. The second principle relates to writing numbers, using a system called “the place value system.” It means that the value of each digit is determined by its position in the number. The rightmost digit enumerates ones, the second digit from the right enumerates tens, the third from the right hundreds, and so forth.

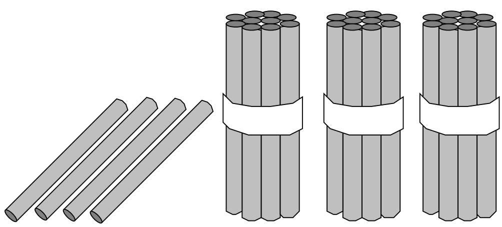
The decimal system is used to organize and represent numbers. It is based on two principles: grouping tens, and ascribing a digit with a value according to its place in the number. The more to the left the digit is placed, the greater the value.

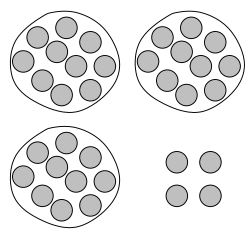
The decimal system simplifies writing large numbers and facilitates doing calculations with them.

Numbers were Not Born Organized

When I tried as an adult to learn to play ping-pong, the instructor discouraged me. He claimed I was too used to the wrong movements, and that it would be difficult to get rid of them. This problem doesn't usually occur in arithmetic, where prior knowledge is generally helpful. But there is one area in which the knowledge a child brings from kindergarten creates a problem: the decimal system. In first grade he must learn anew, in a different way. The reason is that the child already knows how to count, and to him the decimal system is as natural as if it were an integral part of the number. In fact, the same is true of adults. Most of us find it difficult to imagine representing numbers in any different way.

The most important thing to understand about the decimal system is possibly that numbers were not born naturally organized — we are the ones who organize them. Thirty four objects are not naturally clustered into 3 clusters of 10. We are the ones who group them into 3 tens and leave 4 single ones.





Why is this important to know? Because a child who understands that he himself puts together and takes apart the tens, at his own whim, no longer sees calculations using the decimal system as magic. He knows that when ten ones appear in a calculation, he can group them to form a ten. And more importantly, if he knows he is the one who grouped them, he also understands that he can take them apart when the need arises. The need to take apart tens does indeed arise, in subtraction.

This is also the reason why a child must experiment in grouping tens himself, without using instruments that perform the job for him. He must group objects into sets of 10: have him thread 10 beads on a string, draw small circles on a page and ask him to group them into tens and draw a line around each group. If, for example, he drew lines around 4 tens, and 5 circles remained outside the lines, he should understand: There are 45 circles.

Second graders should experiment in grouping hundreds. Ten bundles of ten matches each are grouped to form an element called "a hundred." Don't be afraid of using concrete examples of grouping even in the thousands. Once a child experiences this, at least once, he develops a good sense of the fundamental nature of the decimal system.

Where do We Begin?

When teaching, one should always begin with the familiar. In this case, the familiar is the child's knowledge of counting. He has a vague sense that when crossing the border at 10 some sort of operation is performed — putting aside a ten and starting anew.

This knowledge now needs to be explicitly worded. The process of grouping tens, still vague, must be clarified. One way to do so is to count objects along with the children. After counting ten objects, tie them up together, or set them aside in a pile, and continue counting: eleven, twelve, thirteen ...

Historical Note

Who Invented the Decimal System?

The system we are familiar with today was invented in India. But the priority actually belongs to the ancient Babylonians: They had already invented both principles (grouping and place value) some 3,700 years ago.

However, the Babylonians didn't group tens. They collected sets of 60. That is, 60 ones were grouped to form one element of 60, and 60 elements of 60 were grouped to form a new element, containing (in our notation, of course!) 3,600 ones. This method of writing was adopted by the Greeks who used it for complicated calculations, such as those used in astronomy.

The use of the decimal system began, as mentioned, in India during the 6th or 7th century. The Arabs adopted the system in the 8th century, and introduced it in Europe during the 12th century.

The Arabs also adopted the way of writing digits from India. But the digits developed differently in different places. The Western world uses the digits that evolved in Arabic Spain. The Arabs themselves, on the other hand, use the digits that evolved in the Middle East.

The Story of King Krishna

May the God of History forgive me, but when I teach the decimal system in the second or third grade I tell a story of a king, named Krishna (in honor of the Indians), who loved gold more than anything else. Krishna had a cellar full of gold coins. Every morning he would go down to the cellar to see how many coins he had. But there were so many that it was difficult to estimate their number. Therefore he asked his curator to collect the coins in bags — 10 coins in a bag. Soon enough it became hard to keep count of the bags. So, he asked his curator to collect every ten bags of coins in a sack. When there were too many sacks, he collected every ten sacks in a wooden chest.

Krishna also liked to show off his wealth. Every morning he would mark the number of gold coins in his possession above the palace gate. On the first day, when he had only begun his collection, he had 7 coins. He wrote above the palace gate:

$$7 \odot$$

On the second day, he received 3 more coins. What did he do? He gathered all his coins in a bag of ten and wrote above the gate:

1 ♉

The following day, he received 5 more coins. Now he had a bag and five additional coins. He wrote:

1 ♉ 5 $\odot$

Why did he draw the bag on the left and the coins on the right? Because arithmetic, like English, is written from left to right, and the bag, containing more coins, is more important.

The next day, he had 8 additional coins. He grouped 5 of them with the 5 single coins in an additional bag. He now had two bags, and 3 single coins. Above the gate he wrote:

2 ♉ 3 $\odot$

He continued to do the same each day, until one day his advisor asked: "Why do you bother to draw coins and bags? Everyone already knows that the right digit represents coins and the left represents bags. You can write only the digits!" And so, instead of writing 2 ♉ 3 $\odot$, the king wrote:

2 3

And that is how the decimal system was invented (at least, according to this story).

The Digit 0

On the fourth day, as mentioned, Krishna had 23 coins. On the fifth day, he had 7 more. He grouped the 3 and the 7 into one ten, and so had 3 bags, namely, 3 tens. Proudly, he wrote above the palace gate:

3

He remembered — there was no need to write 3 of what. Everyone understood anyhow! But when he took a second look, he was horrified: How would they know that it didn't mean "3 coins?" He must somehow make it clear that there were 3 bags (that is, three tens)! Something must take the place of the coins, so that it would be obvious that the 3 indicates bags!

And so King Krishna invented (this time, it is the God of Arithmetic I must beg to forgive me for the literary liberty) the digit 0. He wrote 0 in the rightmost place, to indicate that there were no single coins, but that there is a place in the number for coins. And so, he wrote:

3 0

as we are used to writing the number nowadays.

The digit 0 is like the backpack a child places on a chair to declare that the seat is taken. In the number 201, it means: "There are no tens. But there is a place for tens." This is how we know that the 2 indicates hundreds. If the 0 had been omitted, the number would have been 21, where the 2 indicates 2 tens.

Historical Note

The historical truth is that the digit 0, like the principles of grouping and the place value system, was also invented by the Babylonians. However, they did not use it at the end of a number, only in the middle. The difference between 3 and 30 was deduced from context.

Another Advantage of the Decimal System — the Ease of Estimation

Decimal writing enables us to appraise a number with one glance. A brief look at the number 34,522 tells us that it contains 5 digits, and therefore it is in the tens of thousands. The first digit is 3 — therefore, it is approximately 30,000. Like the population of a town. The additional digits, we know, are less significant than the first one.

Store owners take advantage of our habits of estimation: They sell a toy at $ 29.99, knowing that we will pay more attention to the first digit than to the following ones. There is also a psychological aspect: The missing cent is supposedly a discount, and discounts, as is well known, have an attraction beyond their actual value.

What is Learned?

What is Learned in Elementary School?

What mathematical baggage should a child carry out of school? This was one of the first questions I asked myself when I started teaching in elementary school. I had no idea how simple the answer was: a deep understanding of the essence of the number and the four arithmetical operations.

However, this simplicity is misleading. We have just learned that behind the innocent term "the four arithmetical operations" lie two basic principles that are not simple at all: *the meaning* of the operations and the way to *calculate* them. The meaning, as mentioned, is the link to reality. Calculation, on the other hand, means figuring out the decimal representation of the result. Therefore, mastering it requires an in-depth understanding of the decimal system.

From now on say: In elementary school we learn *the meaning of the operations* and the rules derived from it, and *the decimal system.*

Fundamental Structures of Thought

A child does not enter school a blank page. He already knows, or should know, many things. As in life, the important principles are learned at an early age. And as in life, it is the basic principles that are the hardest to pinpoint. We are unaware of most of our basic mechanisms of thought. Everyone knows, for example, that if you climb up 4 steps, you have to climb down 4 steps to get back to the starting point. However, we weren't born with this knowledge. Acquiring it was a true accomplishment.

The following is a list of several structures of thought of which a child starting school has a certain perception. One should remember that the child's grasp of these principles is usually vague and intuitive, and therefore they have to be taught again, this time explicitly.

Left – Right
Up – Down
Large – Small
Before – After, in space
Before – After, in time
Equality (of shapes, or numbers)
Symmetry
Counting Objects

Enumeration

(that is, the ability to repeat the numbers in their correct order)

Reversal

(If you're bigger than me, then I'm smaller than you. If I climbed up 3 steps, to return, I need to climb down 3 steps.)

Grouping Sets

Quite surprisingly, most items on this list deal with relations. Even "left–right" is a relationship: One object can be to the right or to the left of another.

Tip for Parents:

An important asset for a child entering school is knowledge of left–right directions. This is the preamble to the concept of order, and it is essential in all fields of mathematics. Numbers are usually written from left to right, and this is the basis for representing numbers using the decimal system.

In general, the main concepts a child should be familiar with when preparing for the first grade are concepts of relation: "before–after," "up–down," "more–less." These concepts are encountered in any game the child plays, and all a parent needs to do is draw his attention to them. When throwing dice ask — who got more? How much more?

The Curriculum in a Nutshell

The following abridged curriculum may help you understand where your child currently stands, where he's headed, and what to expect. It is important to remember that different textbooks cover a different extent of material, and different schools deviate from the dictated curriculum in various ways. Therefore, the following is an average of kinds. The curriculum is written for Grades 1 to 6, the years covered by this book. I also added some points, derived from my personal opinion that early exposure to concepts is always beneficial. It allows the child periods of incubation between encounters with concepts. For example, first graders should be familiar, at a primary level, with the concept of the fraction and its relationship to division. The decimal fraction can be introduced in the second and third grade, through money.

First grade teaching begins with spatial orientation — right, left, up and down. The children are then taught the concept of the number, counting, the large-small relationship, and the order between numbers. Later, they are taught the meaning of arithmetical operations and the decimal system. As to the meaning of the operations, the Israeli curriculum includes the meaning of addition, subtraction and multiplication (there isn't always enough time for multiplication). Slightly more ambitious curricula also include division, as was the case in Israel during the 1960s. When learning the decimal system, the child should experiment in grouping tens, and understand that more than one ten can be grouped: 30, for instance, is 3 tens. A first grader can understand the meaning of decimal writing, and the role of digits in a number: 23 means 2 tens and 3 single ones. As to the calculation of operations, children should know how to add, subtract and multiply within 20 (that is, the result should not exceed 20). Ambitious curricula include addition and subtraction up to 100. If division is also taught, there is no need to dwell on its calculation. Fractions, such as a half, a quarter and even a third, should be taught on an introductory level. Basic measurements should be encountered — of time and of length. Familiarity with time is important not only because of its practical value, but also because of the connection to fractions, and because the relationship of hours to minutes is similar to the relationship of tens to ones in the decimal system.

In the second grade, the children develop a deeper understanding of the decimal system. They are taught numbers up to 100, and calculate addition and subtraction within the boundary of 100. Ambitious curricula reach 1000. As to multiplication, in my experience it is not hard to teach the entire multiplication table towards the end of the year. Some programs make do with the multiplication table up to multiples of 6. To these I would add: an introduction to the two different meanings of division (see chapter on division) and the concept of remainders in division; furthermore, an introduction to fractions and their relationship to division. It is important to introduce at this stage, a fraction of a group — what is $\frac{1}{2}$ of a class of 30 students? Another topic which should be introduced in this grade is measurements (pounds and ounces, feet and yards).

The third grade is mostly dedicated to the decimal system. The children are taught methods of addition and subtraction within 1000 (at least), and vertical multiplication. The concept of the fraction should be expanded. In addition, the method of calculation of division should be taught. Time measurement is learned in more depth, and the students should know how to convert minutes to hours and vice versa, including fractions of hours. The concept of volume should be taught in a concrete manner, by examining the capacity of various containers.

In the fourth grade, the children are taught calculations of addition, subtraction and multiplication in large numbers (for instance, up to a million), which require an abstract understanding of the principles studied so far, since in large numbers, it is more difficult to rely on intuition. The calculation of division ("long division") should be taught. The concept of the fraction is taught in depth. As to measurements, one reaches the relatively abstract concept of area: What a square centimeter and a square meter are.

The fifth grade is dedicated mostly to simple fractions, including operations with them, and to ratios. Factorization of numbers and the power operation are taught, both being necessary for finding a common denominator. Another subject related to fractions is mixed numbers. In some curricula, decimal fractions are taught.

In the sixth grade, the children study the decimal fractions and learn to perform operations with them. Additional subjects include ratio problems and percentages. Negative numbers can also be taught in the sixth grade — depending on the time available and any introduction that may have been made in previous years. (This book does not include negative numbers, since in most cases they are not taught.)

The Special Role of Division

Careful scrutiny of the curriculum detailed above, and of any other curriculum in the world, reveals a surprising fact: Division has a special status. It is awarded with a greater portion of teaching time than any other operation. The turnabout occurs around the middle of the fourth grade. From this point on, until the end of the sixth grade, the children are taught the meanings of division, ratio problems (which are expressed by division) and the efficient, systematic tool used for discussing division and ratios — the fraction.

Why is division so special? Because the operations of addition and subtraction are too simple to describe the world. When things get complicated, multiplication and division are required. A large part of our world operates according to the principles of proportionality. In elections, for example, the number of mandates each party receives is more or less proportional to the number of votes it received. Proportionality is a guiding principle in understanding our environment, and proportionality is expressed by division.

Another reason for spending more time on division is that it is more difficult than the other operations. Of the four operations, it has the most meanings, it is the hardest to calculate, and the problems it can represent are the most complicated.

Spiraling

Pooh was getting rather tired of that sand-pit, and suspected it of following them about, because whichever direction they started in, they always ended up at it, and each time, as it came through the mist at them, Rabbit said triumphantly, "Now I know where we are!" and Pooh said sadly, "So do I."

A. A. Milne, "The House on Pooh Corner"

Like the characters in "The House on Pooh Corner," when studying arithmetic the same point is repeated over and over. But, unlike in the story, each time we are wiser and view the matter from a different angle. Educators refer to this as "spiral learning." Like a spiral, we pass over the same point again and again. And like a spiral, each time it is at a higher level.

Take, for example, the decimal system. From the vague perception of grouping tens in kindergarten, the first grader moves on to explicit wording of the principle of grouping. He knows how to calculate $8+5$, by grouping a ten from the result and leaving the remaining 3 as ones. Later on he can use the exercise $8 + 5 = 13$ to calculate $28 + 5$, which contains many additional components: breaking 28 into 20 and 8, adding the 8 and the 5 to obtain 13, and joining the components back together as $13 + 20$. A third and fourth grader can generalize the same principle to group tens into hundreds, and hundreds into thousands. In the sixth grade, a bridge is built between fractions and the decimal system, in the form of decimal fractions and percentages.